

# Klein's Fundamental 2-Form of Second Kind for the $C_{ab}$ Curves

Joe SUZUKI

Department of Mathematics, Osaka University, Machikaneyama Toyonaka,  
Osaka 560-0043, Japan

E-mail: [suzuki@math.sci.osaka-u.ac.jp](mailto:suzuki@math.sci.osaka-u.ac.jp)

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**Abstract.** In this paper, we derive the exact formula of Klein's fundamental 2-form of second kind for the so-called  $C_{ab}$  curves. The problem was initially solved by Klein in the 19th century for the hyper-elliptic curves, but little progress had been seen for its extension for more than 100 years. Recently, it has been addressed by several authors, and was solved for subclasses of the  $C_{ab}$  curves whereas they found a way to find its individual solution numerically. The formula gives a standard cohomological basis for the curves, and has many applications in algebraic geometry, physics, and applied mathematics, not just analyzing sigma functions in a general way.

*Key words:*  $C_{ab}$  curves; Klein's fundamental 2-form of second kind; cohomological basis; symmetry

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## 1 Introduction

Since Abel, Jacobi, Poincaré, and Riemann established its framework, theories of Abelian and modular functions associated with algebraic curves have been of crucial importance in algebraic geometry, physics, and applied mathematics. Algebraic curves in this paper are intended as compact Riemann surfaces.

The study of the hyper-elliptic curves goes back to the beginning of the 20th century, and these appear in much detail in advanced text-books such as Baker [2] and Cassels and Flynn [9]. However, little has been considered for more general curves than the hyper-elliptic curves. In this paper, we consider a class of curves ( $C_{ab}$  curves) in the form

$$\sum_{i,j} c_{i,j} x^i y^j = 0, \quad (1.1)$$

where  $(i, j)$  range over  $ai + bj \leq ab$ ,  $0 \leq i \leq b$ ,  $0 \leq j \leq a$  for some mutually prime integers  $a, b$ , and  $c_{i,j}$  are constants such that (1.1) implies  $\sum_{i,j} i c_{i,j} x^{i-1} y^j \neq 0$  or  $\sum_{i,j} j c_{i,j} x^i y^{j-1} \neq 0$ .

This paper studies cohomologies of  $C_{ab}$  curves in terms of which the Klein–Weierstrass construction of multivariate Abelian and sigma functions is made possible. It does not seek further theories but focuses on the description of cohomologies of these curves.

According to the Riemann–Roch theorem, the entire holomorphic differentials make a vector space over  $\mathbb{C}$  of dimension  $g$ . For example, we can take them as

$$du_i(x, y) = \frac{x^{i-1} dx}{2y}, \quad (1.2)$$

$i = 1, 2, \dots, g$  for the hyper-elliptic curve

$$y^2 = \sum_{i=0}^{2g+1} c_i x^i \quad (1.3)$$

with  $c_{i,0} = c_i$ ,  $c_{i,1} = 0$ ,  $i = 0, \dots, 2g+1$ .

We consider a 2-form  $((x, y), (z, w)) \mapsto R((x, y), (z, w)) dx dz$  over  $C \times C$  that has a (double) pole only at  $(z, w) = (x, y)$  and is normalized:

$$\lim_{(z,w) \rightarrow (x,y)} R((x, y), (z, w)) (x - z)^2 = 1.$$

Such a 2-form can be written as

$$R((x, y), (z, w)) := \frac{d}{dz} \Omega((x, y), (z, w)) + \sum_{i=1}^g \frac{du_i(x, y)}{dx} \frac{dr_i(z, w)}{dz}, \quad (1.4)$$

using differentials  $\{dr_i\}_{i=1}^g$  of the second kind that have poles only at infinity and a meromorphic function  $\Omega((x, y), (z, w))$ . If we further require symmetry:

$$R((x, y), (z, w)) = R((z, w), (x, y)),$$

then such an  $\{dr_i\}$  is unique modulo operation of  $\text{Sp}(2g, \mathbb{Z})$  and the space generated by  $\{du_i\}$  (see [5] for hyper-elliptic curves and [8, 11] for general curves). We call such a normalized symmetric  $R((x, y), (z, w)) dx dz$  Klein's fundamental 2-form of second kind [14, 15].

For example, for the hyper-elliptic curve (1.3), such a 2-form can be obtained using the canonical meromorphic differentials  $\{dr_i\}_{i=1}^g$

$$dr_i(z, w) = \sum_{k=i}^{2g-i} c_{k+1+i} (k+1-i) \frac{z^k}{2w} dz \quad (1.5)$$

and 1-form

$$\Omega((x, y), (z, w)) := \frac{y+w}{2(x-z)y} dx.$$

It is known that the normalized symmetric bilinear  $R((x, y), (z, w))$  can be expressed using (1.4) for  $C_{ab}$  [17] and telescopic curves [1] although  $\Omega((x, y), (z, w))$  needs more extensions. However, for the canonical meromorphic differentials  $\{dr_i\}_{i=1}^g$ , no exact formula has been given. For example,  $C_{34}$  and  $C_{35}$  curves and  $C_{45}$  curves were studied in [6] and [12], respectively. Recently, the reference [7] addressed  $C_{3,g+1}$  curves with genus  $g$ . These results dealt with a specific class of curves and failed to obtain the formula for general  $C_{ab}$  curves with arbitrary mutually prime  $a, b$ . As a result, still many researchers are either numerically calculating or algorithmically computing  $\{dr_i\}_{i=1}^g$  given  $\{c_{i,j}\}$  and  $\{du_i\}_{i=1}^g$ . This paper extends (1.5) and obtains the general formula in a closed form for the  $C_{ab}$  curves. It contains all the existing results and has many applications including algebraic expression of sigma functions [17], defining equations of the Jacobian varieties, etc. [5, 10].

This paper is organized as follows: Section 2 sets up the holomorphic differentials  $\{du_i\}_{i=1}^g$  and gives background of this paper. Section 3 states and proves the main result (formula) and gives two typical examples both of which extend the previous cases. In the last section, we raise an open problem.

## 2 Background

Let  $a, b$  be mutually prime positive integers, and  $C$  the curve defined by

$$F(x, y) := \sum_{i,j} c_{i,j} x^i y^j = 0$$

with a unique point  $\mathcal{O}$  at which the zero orders of  $x$  and  $y$  are  $a$  and  $b$ , respectively, where  $(i, j)$  range over

$$D := \{(i, j) \mid ai + bj \leq ab, 0 \leq i \leq b, 0 \leq j \leq a\},$$

and  $c_{i,j}$  are constants such that either  $\sum_{i,j} i c_{i,j} x^{i-1} y^j \neq 0$  or  $\sum_{i,j} j c_{i,j} x^i y^{j-1} \neq 0$  for each  $(x, y)$  on the curve.

We consider the set of 1-forms [17]

$$du_{i,j} = \frac{x^i y^j}{\frac{\partial F}{\partial y}(x, y)} dx,$$

where  $(i, j)$  range over

$$J(a, b) := \{(i, j) \mid i, j \geq 0, ai + bj \leq ab - a - b\}.$$

We know by the general theory that for  $g$  variable points  $(x_1, y_1), \dots, (x_g, y_g)$  on  $C$ , the sum of integrals from  $\mathcal{O}$  to those  $g$  points

$$u = (u_{i,j})_{(i,j) \in J(a,b)} = \sum_{(i,j) \in J(a,b)} \int_{\mathcal{O}}^{(x_i, y_i)} du_{i,j}$$

fill the whole space  $\mathbb{C}^g$ , where the weights of the variables  $u_{i,j}$  are  $ab - a(i+1) - b(j+1)$ . If we regard the weight of each coefficient  $c_{i,j}$  in (1.1) is  $ai + bi - ab$ , the weights assigned to the differentials render  $F(x, y)$  homogeneous.

**Proposition 2.1.** *The set  $J(a, b)$  has*

$$\#J(a, b) = g = \frac{(a-1)(b-1)}{2}$$

*elements, and the zero orders of  $\{du_{i,j}\}$  are nonnegative and different.*

**Definition 2.2.** The 2-form  $R((x, y), (z, w))dx dz$  on  $C \times C$  is called the fundamental 2-form of the second kind if the following conditions are satisfied

- 1) it is symmetric:  $R((x, y), (z, w)) = R((z, w), (x, y))$ ,
- 2) it has its only pole along the diagonal of  $C \times C$ , and
- 3) in the vicinity of each point, it is expanded in power series as

$$R((x, y), (z, w))dx dz = \frac{dt_{xy} dt_{zw}}{(t_{x,y} - t_{z,w})^2} + O(1)$$

as  $(z, w) \rightarrow (x, y)$ , where  $t_{x,y}$  and  $t_{z,w}$  are local coordinates of points  $(x, y)$  and  $(z, w)$ , respectively.

We shall look for a realization of  $R((x, y), (z, w))$  in the form

$$R((x, y), (z, w)) = \frac{G((x, y), (z, w))}{(x - z)^2 \frac{\partial F}{\partial y}(x, y) \frac{\partial F}{\partial w}(z, w)},$$

where  $G((x, y), (z, w))$  is a symmetric polynomial in its variables.

**Example 2.3** (hyper-elliptic curves [3, 4]). For the hyper-elliptic curve (1.3), in which  $a = 2$  and  $b = 2g + 1$ , where  $g$  is the genus of the curve  $C$ , then (1.1) expresses a hyper-elliptic curve, the meromorphic function on  $C \times C$

$$\Omega((x, y), (z, w)) = \frac{y + w}{2y(x - z)}$$

and differentials (1.2), there exists  $\{dr_i\}_{i=1}^g$  such that (1.4) is symmetric. In fact,

$$\begin{aligned} \frac{d}{dz} \Omega((x, y), (z, w)) &= \frac{\frac{dw}{dz}(x - z) + (y + w)}{2y(x - z)^2} = \frac{(x - z) \sum_{k=1}^{2g+1} kc_k z^{k-1} + 2w(y + w)}{4yw(x - z)^2} \\ &= \frac{x \sum_{k=1}^{2g+1} kc_k z^{k-1} - \sum_{k=1}^{2g+1} (k - 2)c_k z^i + 2c_0 + 2yw}{4yw(x - z)^2}. \end{aligned} \quad (2.1)$$

If we add

$$(x - z)^2 \sum_{k=1}^g x^{k-1} \sum_{h=2k+1}^{2g+1} c_h (h - 2k) z^{h-k-1}$$

to the numerator of (2.1), we obtain

$$G((x, y), (z, w)) = 2 \sum_{k: \text{ even}} c_k 2x^{k/2} z^{k/2} + \sum_{k: \text{ odd}} c_k \left( x^{\frac{k+1}{2}} z^{\frac{k-1}{2}} + x^{\frac{k-1}{2}} z^{\frac{k+1}{2}} \right) + 2c_0 + 2yw,$$

where

$$\begin{aligned} &-(k - 2)z^k + kz^{k-1}x + (x - z)^2 \sum_{h=1}^{i/2} (k - 2h)z^{k-h-1}x^{h-1} \\ &= -(k - 2l)z^{k-l+1}x^{l-1} + (k - 2l + 2)z^{k-l}x^l + (x - z)^2 \sum_{h=l}^{\lfloor k/2 \rfloor} (k - 2h)z^{k-h-1}x^{h-1} \end{aligned}$$

for  $l = 1, 2, \dots, \lfloor i/2 \rfloor$  has been applied ( $\lfloor k/2 \rfloor = k/2$  and  $(k - 1)/2$  when  $k$  is even and odd, respectively), which means that by choosing

$$\frac{dr_i}{dz}(z, w) = \sum_{h=2i+1}^{2g+1} \frac{(h - 2i)z^{h-i-1}}{2w},$$

we obtain

$$R((x, y), (z, w)) = R((z, w), (x, y))$$

and

$$\lim_{(z, w) \rightarrow (x, y)} R((x, y), (z, w))(x - z)^2 = \lim_{(z, w) \rightarrow (x, y)} \frac{G((x, y), (z, w))}{4yw} = 1.$$

### 3 The fundamental 2-form of the second kind for the $C_{ab}$ curves

Hereafter, we denote  $F_z := \frac{\partial F(z,w)}{\partial z}$ ,  $F_w := \frac{\partial F(z,w)}{\partial w}$ ,  $H := \frac{F(y,z)-F(w,z)}{y-z}$ ,  $H_z := \frac{\partial H}{\partial z}$ , and  $H_w := \frac{\partial H}{\partial w}$ .

We find symmetric  $R((x,y),(z,w))$  for the meromorphic function

$$\Omega((x,y),(z,w)) := \frac{H}{(x-z)F_y}.$$

To this end, if we note  $\frac{dw}{dz} = -\frac{F_z}{F_w}$ , we have

$$\frac{d\Omega}{dz} = \frac{(H_z + H_w \frac{dw}{dz})(x-z) + H}{(x-z)^2 F_y} = \frac{(H_z F_w - H_w F_z)(x-z) + H F_w}{(x-z)^2 F_y F_w}. \quad (3.1)$$

Let  $g_j := \sum_{i: 0 \leq ai+bj \leq ab} c_{i,j} z^i$ ,  $g'_j := \frac{\partial g_j}{\partial z}$ , and  $h_j := \sum_{i=0}^{j-1} w^i y^{j-1-i}$  for  $j = 0, \dots, a$ . Then, one checks

$$F_w = \sum_{j=0}^a j w^{j-1} g_j, \quad H = \sum_{j=0}^a h_j g_j, \quad H_z = \sum_{j=0}^a h_j g'_j, \quad H_w = \sum_{j=0}^a \frac{\partial h_j}{\partial w} g_j. \quad (3.2)$$

Let

$$I(i,j) := \left( -w^i \frac{\partial h_j}{\partial w} + j w^{j-1} h_i - \frac{\partial h_i}{\partial y} w^j \right) g'_i g_j (x-z) + (j w^{j-1} h_i - j h_{j-1} w^i) g_i g_j$$

for  $i, j = 0, \dots, a$ . Then, from (3.2) and that the arithmetic is modulo  $F(z,w) = \sum_{j=0}^a w^j g_j = 0$ , we have

$$\begin{aligned} \sum_{i=0}^a \sum_{j=0}^a I(i,j) &= \sum_{i=0}^a \left\{ -w^i g'_i H_w + h_i g'_i F_w - \frac{\partial h_i}{\partial y} g'_i \sum_{j=0}^a w^j g_j \right\} (x-z) \\ &\quad + \sum_{i=0}^a \left\{ h_i g_i F_w - w^i g_i \sum_{j=0}^a j h_{j-1} g_j \right\}, \end{aligned}$$

which coincides with the numerator of (3.1).

We seek  $\{\frac{dr_{i,j}}{dz}\}_{(i,j) \in J(a,b)}$  such that the 2-form

$$R((x,y),(z,w)) := \frac{d\Omega(z,w)}{dz} + \sum_{(i,j) \in J(a,b)} \frac{du_{i,j}}{dx} \frac{dr_{i,j}}{dz}$$

is symmetric, in other words, its numerator

$$\sum_{u=0}^a \sum_{v=0}^a I(u,v) + (x-z)^2 \sum_{(i,j) \in J(a,b)} u_{i,j}(x,y) r_{i,j}(z,w)$$

is symmetric, where

$$\frac{dr_{i,j}}{dz} = \frac{r_{i,j}(z,w)}{F_w}$$

is the differential of the second kind given the differentials of the first kind

$$\frac{du_{i,j}}{dx} = \frac{u_{i,j}(x,y)}{F_y}$$

with  $u_{i,j}(x,y) = x^i y^j$ .

Let  $\underline{\text{mod}}(\alpha, \beta) := \alpha - \lfloor \alpha/\beta \rfloor * \beta$ , and  $\overline{\text{mod}}(\alpha, \beta) := \begin{cases} \underline{\text{mod}}(\alpha, \beta) & \underline{\text{mod}}(\alpha, \beta) \neq 0, \\ \alpha & \underline{\text{mod}}(\alpha, \beta) = 0 \end{cases}$  for positive

integers  $\alpha, \beta$ , and  $\lceil \gamma \rceil$  and  $\lfloor \gamma \rfloor$  are the smallest integer no less than  $\gamma$  and the largest integer no more than  $\gamma$ , respectively for positive real  $\gamma$ . For example,  $\overline{\text{mod}}(3, 2) = \underline{\text{mod}}(3, 2) = 2$ ,  $2 = \overline{\text{mod}}(4, 2) \neq \underline{\text{mod}}(4, 2) = 0$ , etc.

Then, the following property plays an important role in the derivation of the main theorem:

**Proposition 3.1.** *If  $m \leq n$ , then*

$$\begin{aligned} I(m, n) + I(n, m) &= mw^{m-1}y^{n-1}g_m g'_n(x-z) + mw^{m-1}y^{n-1}g_m g_n \\ &\quad + mw^{n-1}y^{m-1}g'_m g_n(x-z) + mw^{n-1}y^{m-1}g_m g_n \\ &\quad - \sum_{k=m+1}^{n-1} [(n-k)g_m g'_n + (k-m)g'_m g_n](x-z) + (n-m)g_m g_n \\ &\quad \times w^{k-1}y^{m+n-k-1}. \end{aligned} \quad (3.3)$$

**Theorem 3.2.**  *$R((x, y), (z, w))$  is symmetric when*

$$r_{i,j}(z, w) = \sum_{u \leq j} \sum_r \sum_s c_{r,u} c_{s,j+1} u(s-i-1) z^{r+s-i-2} w^{u-1} \quad (3.4)$$

$$+ \sum_{j+1 \leq v} \sum_r \sum_s c_{r,j+1} c_{s,v} (j+1)(r-i-1) z^{r+s-i-2} w^{v-1} \quad (3.5)$$

$$\begin{aligned} &+ \sum_{u \leq j} \sum_{j+2 \leq v} \sum_r \sum_s c_{r,u} c_{s,v} \{(i+1)(v-u) - (j+1-u)s \\ &\quad - (v-j-1)r\} z^{r+s-i-2} w^{u+v-j-2}, \end{aligned} \quad (3.6)$$

where  $(r, s)$  range over  $s \geq i+2$  in  $D$  for the first term, over  $r \geq i+2$  in  $D$  for the second term, and over  $(j+1-u)s + (v-j-1)r \leq (i+1)(v-u)$  and  $r+s \geq i+2$  in  $D$  for the last term.

The symmetric value is given by  $R((x, y), (z, w)) = \frac{G((x, y), (z, w))}{(x-z)^2 \frac{\partial F(x, y)}{\partial x} \frac{\partial F(z, w)}{\partial z}}$  with

$$\begin{aligned} G((x, y), (z, w)) &= \sum_{u=0}^a G_{u,u}^{(12)} + \sum_{u=0}^a \sum_{u < v} \{G_{u,v}^{(12)} + G_{u,v}^{(34)} + G_{u,v}^{(5)}\}, \\ G_{u,v}^{(12)} &:= u \sum_r \sum_s c_{r,u} c_{s,v} x^r z^s y^{u-1} w^{v-1}, \end{aligned} \quad (3.7)$$

$$G_{u,v}^{(34)} := u \sum_r \sum_s c_{r,u} c_{s,v} x^s z^r y^{v-1} w^{u-1}, \quad (3.8)$$

and

$$\begin{aligned} G_{u,v}^{(5)} &:= - \sum_{k=u+1}^{v-1} \sum_r \sum_s c_{r,u} c_{s,v} \{\overline{\text{mod}}\{(k-u)s + (v-k)r, v-u\} z^{\lceil \frac{(k-u)s + (v-k)r}{v-u} \rceil} \\ &\quad \times x^{\lfloor \frac{(k-u)r + (v-k)s}{v-u} \rfloor} + \underline{\text{mod}}\{(k-u)r + (v-k)s, v-u\} z^{\lfloor \frac{(k-u)s + (v-k)r}{v-u} \rfloor} \\ &\quad \times x^{\lceil \frac{(k-u)r + (v-k)s}{v-u} \rceil}\} w^{k-1} y^{u+v-k-1}. \end{aligned} \quad (3.9)$$

**Proof.** First of all, we prove

$$u \sum_r \sum_s c_{r,u} c_{s,v} \{t^{(12)}(x, z) + (x - z)^2 \Delta^{(12)}(x, z)\} w^{u-1} y^{v-1} = G_{u,v}^{(12)}$$

for  $0 \leq u \leq v \leq a$ , where

$$t^{(12)}(x, z) := sz^{r+s-1}(x - z) + z^{r+s}$$

and

$$\Delta^{(12)}(x, z) := \sum_{k=1}^{s-1} kz^{r+k-1}x^{s-k-1}.$$

In fact, we see

$$\begin{aligned} t^{(12)}(x, z) + (x - z)^2 \Delta^{(12)}(x, z) \\ = -(s - l)z^{r+s-l+1}x^{l-1} + (s - l + 1)z^{r+s-l}x^l + (x - z)^2 \sum_{k=1}^{s-l} kz^{r+k-1}x^{s-k-1} \end{aligned}$$

for  $l = 1, \dots, s$ . In particular,

$$\begin{aligned} \sum_r \sum_s u c_{r,u} c_{s,v} \Delta^{(12)}(x, z) y^{v-1} w^{u-1} \\ = \sum_{(i,j) \in J(a,b)} x^i y^j \sum_r \sum_s u c_{r,u} c_{s,j+1} (s - i - 1) z^{r+s-i-2} w^{u-1}, \end{aligned}$$

where  $(r, s)$  range over  $s \geq i + 1$  in  $D$ . Thus, from the first and second terms in (3.3), we obtain (3.4) and (3.7).

Similarly,

$$u \sum_r \sum_s c_{r,u} c_{s,v} \{t^{(34)}(x, z) + (x - z)^2 \Delta^{(34)}(x, z)\} w^{v-1} y^{u-1} = G_{u,v}^{(34)}$$

for  $0 \leq u \leq v \leq a$ ,

$$t^{(34)}(x, z) := rz^{r+s-1}(x - z) + z^{r+s}, \quad \Delta^{(34)}(x, z) := \sum_{k=1}^{r-1} kz^{s+k-1}x^{r-k-1},$$

and

$$\begin{aligned} \sum_r \sum_s u c_{r,u} c_{s,v} \Delta^{(34)}(x, z) y^{u-1} w^{v-1} \\ = \sum_{(i,j) \in J(a,b)} x^i y^j \sum_r \sum_s u c_{r,j+1} c_{s,v} (r - i - 1) z^{r+s-i-2} w^{v-1}, \end{aligned}$$

where  $(r, s)$  range over  $r \geq i + 1$  in  $D$ . Thus, from the third and fourth terms in (3.3), we obtain (3.5) and (3.8).

On the other hand, we claim

$$\begin{aligned} t_k^{(5)}(x, z) + (x - z)^2 \Delta_k^{(5)}(x, z) = -\{(p + 1)(v - u) - (v - k)s - (k - u)r\} z^{r+s-p} x^p \\ - \{(v - k)s + (k - u)r - p(v - u)\} z^{r+s-p-1} x^{p+1} \end{aligned} \quad (3.10)$$

for  $0 \leq u \leq v \leq a$ , where

$$\begin{aligned} t_k^{(5)}(x, z) &:= \{s(v-k) + r(k-u)\}z^{r+s-1}(x-z) + (v-u)z^{r+s}, \\ \Delta_k^{(5)}(x, z) &:= (x-z)^2 \sum_{h=1}^p \{(v-k)s + (k-u)r - (v-u)h\}z^{r+s-h-1}x^{h-1}, \end{aligned}$$

where  $p$  is a unique integer such that

$$p(v-u) \leq (v-k)s + (k-u)r < (p+1)(v-u). \quad (3.11)$$

In fact, we see

$$\begin{aligned} t_k^{(5)}(x, z) + (x-z)^2 \Delta_k^{(5)}(x, z) &= -\{s(v-k) + r(k-u) - l(v-u)\}z^{r+s-l+1}x^{l-1} \\ &\quad + \{s(v-k) + r(k-u) - (l-1)(v-u)\}z^{r+s-l}x^l \\ &\quad + (x-z)^2 \sum_{h=l}^p \{(v-k)s + (k-u)r - (v-u)h\}z^{r+s-h-1}x^{h-1} \end{aligned}$$

for  $l = 1, 2, \dots, p+1$ .

Similarly, we obtain

$$\begin{aligned} t_{u+v-k}^{(5)}(x, z) + (x-z)^2 \Delta_{u+v-k}^{(5)}(x, z) &= -\{(q+1)(v-u) - (v-k)r - (k-u)s\}z^{r+s-q}x^q \\ &\quad - \{(v-k)r + (k-u)s - q(v-u)\}z^{r+s-q-1}x^{q+1} \end{aligned} \quad (3.12)$$

for  $0 \leq u \leq v \leq a$ , where

$$\begin{aligned} t_{u+v-k}^{(5)}(x, z) &:= \{r(v-k) + s(k-u)\}z^{r+s-1}(x-z) + (v-u)z^{r+s}, \\ \Delta_{u+v-k}^{(5)}(x, z) &:= (x-z)^2 \sum_{h=1}^p \{(v-k)r + (k-u)s - (v-u)h\}z^{r+s-h-1}x^{h-1}, \end{aligned}$$

where  $q$  is a unique integer such that

$$q(v-u) \leq (v-k)r + (k-u)s < (q+1)(v-u). \quad (3.13)$$

From (3.11) and (3.13), we have two possibilities:  $p+q = r+s$  and  $p+q+1 = r+s$ . For the former case, (3.10) and (3.12) are  $-(v-u)z^q x^p$  and  $-(v-u)z^p x^q$ , respectively; and for the latter case, (3.10) and (3.12) are

$$-\{(v-k)s + (k-u)r - q(v-u)\}z^{q+1}x^p + \{(v-k)s + (k-u)r - p(v-u)\}z^q x^{p+1}$$

and

$$-\{(v-k)r + (k-u)s - p(v-u)\}z^{p+1}x^q + \{(v-k)r + (k-u)s - q(v-u)\}z^p x^{q+1},$$

respectively.

Since

$$\begin{aligned} p &= \left\lfloor \frac{(v-k)s + (k-u)r}{v-u} \right\rfloor, & q &= \left\lfloor \frac{(v-k)r + (k-u)s}{v-u} \right\rfloor, \\ (v-k)s + (k-u)r - p(v-u) &= \text{mod}\{(v-k)s + (k-u)r, v-u\}, \\ (v-k)r + (k-u)s - q(v-u) &= \overline{\text{mod}}\{(v-k)r + (k-u)s, v-u\}, \end{aligned}$$



we have (3.9), which means we obtain (3.6) as well

$$\begin{aligned} & \sum_{k=u+1}^{v-1} \sum_r \sum_s c_{r,u} c_{s,v} \Delta_k^{(5)}(x, z) y^{u+v-k-1} w^{k-1} \\ &= \sum_{(i,j) \in J(a,b)} x^i y^j \sum_r \sum_s u c_{r,j+1} c_{s,v} \{(i+1)(v-u) - (j+1-u)s \\ &\quad - (v-j-1)r\} z^{r+s-i-2} w^{v-1}, \end{aligned}$$

where we have chosen for each  $(i, j) \in J(a, b)$ ,  $i = h-1$  and  $j = u+v-k-1$ , so that  $1 \leq i+1 \leq p$  and  $u+1 \leq u+v-j-1 \leq v-1$ . Hence,  $(u, v)$  and  $(s, r)$  need to satisfy

$$u \leq j, \quad j+2 \leq v, \quad (i+1)(v-u) \geq (v-k)s + (k-u)r,$$

respectively. This completes the proof.  $\blacksquare$

**Corollary 3.3.**

$$\lim_{(z,w) \rightarrow (x,y)} R((z, w), (x, y))(x-z)^2 = 1.$$

**Proof.** Let  $f_u := \sum_{0 \leq ar+bu \leq ab, r \geq 0} c_{r,u} x^r$ . Then,  $G_{u,v}^{(12)}$ ,  $G_{u,v}^{(34)}$ , and  $G_{u,v}^{(5)}$  converge to  $u f_u f_v$ ,  $u f_u f_v$ , and  $-(v-u)(v-u-1)y^{u+v-2}$ , respectively. Thus,  $G((x, y), (z, w))$  converges to

$$\begin{aligned} & \sum_{u=0}^a u f_u^2 y^{2u-2} + \sum_{u=0}^a \sum_{u < v} 2u f_u f_v y^{u+v-2} - \sum_{u=0}^a \sum_{u < v} (v-u)(v-u-1) f_u f_v y^{u+v-2} \\ &= \sum_{u=0}^a u f_u^2 y^{2u-2} + \sum_{u=0}^a \sum_{u < v} \{(u+v) - (u-v)^2\} f_u f_v y^{u+v-2} \\ &= \frac{1}{2} \sum_{u=0}^a \sum_{v=0}^a \{u+v - u^2 - v^2 + 2uv\} f_u f_v y^{u+v-2} \\ &= \sum_{u=0}^a \sum_{v=0}^a u v f_u f_v y^{u+v-2} = \sum_{u=1}^a u f_u y^{u-1} \sum_{v=1}^a v f_v y^{v-1} = \left\{ \sum_{u=1}^a u f_u y^{u-1} \right\}^2 = F_y^2, \end{aligned}$$

where

$$\sum_{u=0}^a \sum_{v=0}^a u f_u f_v y^{u+v-2} = \sum_{u=1}^a y^{u-2} \sum_{v=0}^a f_v y^v = 0$$

and

$$\sum_{u=0}^a \sum_{v=0}^a v f_u f_v y^{u+v-2} = \sum_{u=0}^a \sum_{v=0}^a u^2 f_u f_v y^{u+v-2} = \sum_{u=0}^a \sum_{v=0}^a v^2 f_u f_v y^{u+v-2} = 0$$

have been applied. On the other hand, the denominator of  $R((z, w), (x, y))(x-z)^2$  converges to  $F_y^2$  as well. This completes the proof.  $\blacksquare$

**Example 3.4** (generalized hyper-elliptic curves). For the curve  $F(x, y) = y^2 + y f_1 + f_0 = 0$ ,

$$\begin{aligned} G((x, y), (z, w)) &= 2yw + f_1 w + g_1 y + f_1 g_1 \\ &\quad - \sum_{r: \text{ odd}} c_{r,0} \left( z^{\frac{r+1}{2}} x^{\frac{r-1}{2}} + z^{\frac{r-1}{2}} x^{\frac{r+1}{2}} \right) - \sum_{r: \text{ even}} 2c_{r,0} z^{\frac{r}{2}} x^{\frac{r}{2}}, \end{aligned}$$

when

$$\begin{aligned} r_{i,0}(z, w) &= \sum_{r \geq i+2} \sum_{s \geq 0} c_{r,1} c_{s,1} (r-i-1) z^{r+s-i-2} \\ &\quad + \sum_{r \geq i+2} c_{r,1} (r-i-1) z^{r-i-2} + \sum_{r=i+2}^{2i+2} c_{r,0} (2i+2-r) \end{aligned}$$

and converges to

$$2y^2 + 2f_1y + f_1^2 + 2(f_1y + y^2) = (2y + f_1)^2 = F_y^2$$

as  $(z, w) \rightarrow (x, y)$ .

**Example 3.5** (cyclic curves, super-elliptic curves [13]). For the curve  $F(x, y) = y^a + f_0 = 0$ ,

$$\begin{aligned} G((x, y), (z, w)) &= - \sum_r c_{r,0} \sum_{k=1}^{a-1} \left\{ \overline{\text{mod}}((a-k)r, a) z^{\lceil \frac{(a-k)r}{a} \rceil} x^{\lfloor \frac{kr}{a} \rfloor} \right. \\ &\quad \left. + \underline{\text{mod}}(kr, a) z^{\lfloor \frac{(a-k)r}{a} \rfloor} x^{\lceil \frac{kr}{a} \rceil} \right\} w^{k-1} y^{a-k-1}, \end{aligned}$$

when

$$r_{i,j}(z, w) = - \sum_{r=i+2}^b c_{r,0} (ar - a - r - ai - rj) z^{r-2-i} w^{a-2-j}$$

and converges to

$$-2f_0y^{a-2} = F_y^2$$

as  $(z, w) \rightarrow (x, y)$ .

**Example 3.6** (trigonal curves [6, 7]). For the curve  $F(x, y) = y^3 - q(x)y - p(x)$  with  $p(x) =$

$$- \sum_{r=0}^{g+1} c_{r,0} x^r \text{ and } q(x) = - \sum_{r=0}^{\lfloor \frac{2g+2}{3} \rfloor} c_{r,1} x^r,$$

$$\begin{aligned} G((x, y), (z, w)) &= \sum_r c_{r,1} x^r \sum_s c_{s,1} z^s + \sum_r c_{r,1} x^r w^2 + \sum_s c_{s,1} z^s y^2 + 3y^2 w^2 \\ &\quad - \sum_r c_{r,0} \left\{ \overline{\text{mod}}(2r, 3) z^{\lceil 2r/3 \rceil} x^{\lfloor r/3 \rfloor} + \underline{\text{mod}}(r, 3) z^{\lfloor 2r/3 \rfloor} x^{\lceil r/3 \rceil} \right\} y \\ &\quad - \sum_r c_{r,0} \left\{ \overline{\text{mod}}(r, 3) z^{\lceil r/3 \rceil} x^{\lfloor 2r/3 \rfloor} + \underline{\text{mod}}(2r, 3) z^{\lfloor r/3 \rfloor} x^{\lceil 2r/3 \rceil} \right\} w \\ &\quad - \sum_r c_{r,1} \left\{ \overline{\text{mod}}(r, 2) z^{\lceil r/2 \rceil} x^{\lfloor r/2 \rfloor} + \underline{\text{mod}}(r, 2) z^{\lfloor r/2 \rfloor} x^{\lceil r/2 \rceil} \right\} wy \\ &= \sum_r c_{r,1} x^r \sum_s c_{s,1} z^s + \sum_r c_{r,1} x^r w^2 + \sum_s c_{s,1} z^s y^2 + 3y^2 w^2 \\ &\quad - \sum_{r=3m} c_{r,0} \left\{ 3z^{2r/3} x^{r/3} y + 3z^{2r/3} x^{r/3} w \right\} \\ &\quad - \sum_{r=3m+1} c_{r,0} \left\{ 2z^{(2r+1)/3} x^{(r-1)/3} y + z^{(2r-2)/3} x^{(r+2)/3} y \right. \\ &\quad \left. + z^{(r+2)/3} x^{(2r-2)/3} w + 2z^{(r-1)/3} x^{(2r+1)/3} w \right\} \end{aligned}$$

$$\begin{aligned}
& - \sum_{r=3m+2} c_{r,0} \{ z^{(2r+2)/3} x^{(r-2)/3} y + 2z^{(2r-1)/3} x^{(r+1)/3} y \\
& \quad + 2z^{(r+1)/3} x^{(2r-1)/3} w + z^{(r-2)/3} x^{(2r+2)/3} w \} \\
& - 2 \sum_{r=2l} c_{r,1} z^{r/2} x^{r/2} w y \\
& - \sum_{r=2l+1} c_{r,1} \{ z^{(r+1)/2} x^{(r-1)/2} + z^{(r-1)/2} x^{(r+1)/2} \} w y,
\end{aligned}$$

when

$$\begin{aligned}
r_{i,0}(z, w) &= \sum_{r \geq 2i+2} \sum_s c_{r,1} c_{s,1} (r-i-1) z^{r+s-i-2} + \sum_{r \geq 2i+2} c_{r,1} (r-i-1) z^{r-i-2} w^2 \\
& + \sum_{r=i+2}^{\lfloor 3(i+1)/2 \rfloor} c_{r,0} (3i+3-2r) z^{r-i-2} w
\end{aligned}$$

and

$$r_{i,1}(z, w) = \sum_{r=i+2}^{3i+3} c_{r,0} (3i+3-r) z^{r-i-2} + \sum_{r=i+2}^{2i+2} c_{r,1} (2i+2-r) z^{r-i-2} w.$$

In this sense, Theorem 3.2 contains the same formula [7, p. 203] as a special case that was described as

$$\begin{aligned}
r_{i,0}(z, w) &= -w^2 \partial_z D_z^{i+1} q(z) + w z^{\lfloor i/2 \rfloor} D_z^{\lfloor 3i/2+2 \rfloor} \{ 2z \partial_z p(z) - 3(i+1)p(z) \} \\
& + z^i D_z^{2i+2} \left\{ \frac{1}{2} z \partial q(z)^2 - (i+1)q(z)^2 \right\}
\end{aligned}$$

and

$$r_{i,1}(z, w) = w z^{i+1} D_z^{2i+3} \{ z \partial_z q(z) - 2(i+1)q(z) \} + z^{2i+2} D_z^{3i+4} \{ z \partial_z p(z) - 3(i+1)p(z) \},$$

where  $D_s p(z)$  denotes  $\sum_{k=s}^n p_k z^{k-s}$  for polynomial  $p(z) = \sum_{k=0}^n p_k z^k$ .

All of these applications of the general formula coincide with the known results [6, 7].

## 4 Open problem

Let  $F$  be a function field over  $\mathbb{C}$ , and  $\mathcal{O}$  any place of degree one. We define the set

$$L := \{ f \in F \mid \text{ord}_{\mathcal{O}}(f) \geq 0, Q \neq \mathcal{O} \} \cup \{0\}$$

of functions with poles only at  $\mathcal{O}$  and the monoid  $M := \{-\text{ord}_{\mathcal{O}}(f) \mid f \in L\}$ . We choose generators  $a_1, \dots, a_m$  of  $M$  and functions  $x_1, \dots, x_m \in L$  such that  $-\text{ord}_{\mathcal{O}}(x_i) = a_i$ ,  $i = 1, \dots, m$ , where  $a_1, \dots, a_m$  are assumed to be mutually prime. For the homomorphism from the polynomial ring  $\mathbb{C}[X_1, \dots, X_m]$  to the ring  $\mathbb{C}[x_1, \dots, x_m]$  generated by  $x_1, \dots, x_m$ , we can regard a set of generators of the kernel as the defining equations of the curve  $C$  (Miura [16]). The kernel contains  $m$  variables and at least  $m-1$  equations. Let  $d_i$  be the GCM of  $a_1, \dots, a_i$ . The sequence  $a_1, \dots, a_m$  is referred to as telescopic if

$$\frac{a_i}{d_i} \in \left\langle \frac{a_1}{d_{i-1}}, \dots, \frac{a_{i-1}}{d_{i-1}} \right\rangle$$

for  $i = 2, \dots, m$ . Suzuki [18] proved that the number of equations is  $m-1$  if and only if the sequence can be telescopic by permutating the order of the sequence.

This paper considered only the case  $m = 2$ . Our future work contains solving the problem for telescopic and general Miura curves.

## A Proof of Proposition 2.1

For the first part, without loss of generality, we assume  $a < b$ . For each  $0 \leq i \leq b-1$ , let  $f(i)$  be the number of integers  $j$ 's such that

$$j \leq a-1 - \frac{a}{b}(i+1).$$

Since

$$f(0) + f(b-2) = \left\lfloor a-1 - \frac{a}{b} \right\rfloor + 1 + 0 = a-1 = f(i) + f(b-i-2)$$

for  $i = 0, \dots, b-2$ , and  $f(b-1) = 0$ , where  $\lfloor x \rfloor$  is the largest integer no more than  $x$ , we have

$$\sum_{i=0}^{b-1} f(i) = \frac{1}{2} \sum_{i=0}^{b-2} \{f(i) + f(b-2-i)\} + f(b-1) = \frac{(a-1)(b-1)}{2},$$

which means  $\#J(a, b) = \frac{(a-1)(b-1)}{2}$ .

For the latter part, since

$$\text{ord}_{\mathcal{O}}(dr_{i,j}) = \text{ord}_{\mathcal{O}} \left( \frac{x^i y^j}{\sum_{k=1}^a k y^{k-1} f_k} dx \right) = -(ai + bj) + ab - a - b \geq 0$$

for  $(i, j) \in J(a, b)$ , each  $dr_{i,j}$  is holomorphic. On the other hand, since  $\text{ord}_{\mathcal{O}}(dr_{i,j}) \neq \text{ord}_{\mathcal{O}}(dr_{i',j'})$  for  $(i, j) \neq (i', j')$ , the generators  $\{dr_{i,j}\}$  consisting of  $g$  differentials are linearly independent and cover the vector space.

## B Proof of Proposition 3.1

From

$$\begin{aligned} & -w^i \frac{\partial h_j}{\partial w} + j w^{j-1} h_i - \frac{\partial h_i}{\partial y} w^j - j w^{j-1} y^{i-1} \\ &= - \sum_{k=1}^{j-1} k w^{i+k-1} y^{j-1-k} + j \sum_{k=0}^{i-1} w^{j+k-1} y^{i-1-k} - \sum_{k=1}^{i-1} (i-k) w^{j+k-1} y^{i-k-1} - j w^{j-1} y^{i-1} \\ &= - \sum_{k=i+1}^{i+j-1} (k-i) w^{k-1} y^{i+j-k-1} + \sum_{k=j+1}^{i+j-1} (k-i) w^{k-1} y^{i+j-k-1} \\ &= \begin{cases} \sum_{k=j+1}^i (k-i) w^{k-1} y^{i+j-k-1}, & i \geq j+1, \\ 0, & i = j, \\ - \sum_{k=i+1}^j (k-i) w^{k-1} y^{i+j-k-1}, & i \leq j-1, \end{cases} \end{aligned}$$

and

$$j w^{j-1} h_i - j w^i h_{j-1} - j w^{j-1} y^{i-1} = \begin{cases} j \sum_{k=j+1}^i w^{k-1} y^{i+j-1-k}, & i \geq j+1, \\ 0, & i = j, \\ -j \sum_{k=i+1}^j w^{k-1} y^{i+j-1-k}, & i \leq j-1, \end{cases}$$

we have

$$I(i, j) - \{jw^{j-1}y^{i-1}g'_i g_j(x-z) + jw^{j-1}y^{i-1}g_i g_j\} \\ = \begin{cases} \sum_{k=j+1}^{i-1} g_j \{(k-i)g'_i(x-z) + jg_i\} w^{k-1} y^{i+j-k-1}, & i \geq j+1, \\ 0, & i = j, \\ - \sum_{k=i+1}^{j-1} g_j \{(k-i)g'_i(x-z) + jg_i\} w^{k-1} y^{i+j-k-1}, & i \leq j-1. \end{cases}$$

Since  $m \leq n$ , we have  $I(m, n)$  with  $i = n$  and  $j = m$  for  $i \geq j+1$  and  $I(m, n)$  with  $i = m$  and  $j = n$  for  $i \leq j-1$ , so that (3.3) is obtained.

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